11/20/23 **Factorization / Chiral Algebra** Speaker: Lie Qian
Notes: Nikolay

$X$ smooth proper curve

## §1. Factorization Alg / Mod

**Want:** A qcoh sheaf over the space of legs of $X$ (# of legs vary!)

$\mathrm{Ran}(X) = \{$ nonempty finite subsets of $X\} = \operatorname{colim}_I X^I$

top. space

**Prop** If $X$ connected, $\mathrm{Ran}(X)$ is weakly contractible.

Alg. analogue? Factorization Alg & Factorization homology.

**Def** (1) A qcoh sheaf on $\mathrm{Ran}(X)$ is collection of qcoh sheaves $\mathcal{F}_I$ on $X^I$ for all $I$ w/ isoms $\forall\, J \overset{\mathcal{I}}{\twoheadrightarrow} I$, ($\rightsquigarrow \Delta^{J/I}: X^I \to X^J$)

$v^{(\mathcal{I})}: (\Delta^{J/I})^* \mathcal{F}_J \xrightarrow{\sim} \mathcal{F}_I$, compatible w/ composition.

Moreover, require $\mathcal{F}^{(2)}$ have no sections supp. on diagonal

(2) Non-unital factorization alg. $\mathcal{B}$ is a qcoh sheaf on $\mathrm{Ran}(X)$ w/ isom $c_\alpha: j_\alpha^*\left(\boxtimes \mathcal{B}^{(I_i)}\right) \to j_\alpha^* \mathcal{B}^{(I)}$

for a partition $I = I_1 \sqcup \cdots \sqcup I_n$, $j_\alpha: U \hookrightarrow X^I$,

$U = \{x_i \neq x_j\}$.

Ex $\mathcal{F}_I = \mathcal{O}_{X^I}$.

(3) (Unital) Factorization alg $\mathcal{B}$ is a non-unital Fac. alg. & a map of fac. alg. $\mathcal{O} \to \mathcal{B}$ s.t. $\forall\, b \in \mathcal{B}^{(1)}$,

$1 \boxtimes b \in j_* j^* (\mathcal{B}^{(1)} \boxtimes \mathcal{B}^{(1)})$ uniquely extends in

$\mathcal{B}^{(2)} \subseteq j_* j^* (\mathcal{B}^{(1)} \boxtimes \mathcal{B}^{(1)})$ s.t. $\Delta^*(1 \boxtimes b) = b \in \mathcal{B}^{(1)}$.

**Rmk** (1) We have maps $\mathcal{B}^{(I_1)} \boxtimes \mathcal{O}_{X^{(I_2)}} \longrightarrow \mathcal{B}^{(I_1 \sqcup I_2)}$

$\forall f: I_2 \to I_1 \rightsquigarrow \mathcal{I}: I_1 \sqcup I_2 \xrightarrow{(\mathrm{Id}, f)} I_1$, satisfies

$\Delta^*_{\pi}(b \boxtimes 1) = b \in B^{(I,)}$.

(2) Any Fac. alg admits a D-mod. structure $\forall B^I$.

(3) $B_1$ & $B_2$ factorization, can set their tensor product $B$ s.t

$\Rightarrow$ Fact. alg forms monoidal cat.

$$B^{(I)} = B_1^{(I)} \underset{\mathcal{O}_{X^I}}{\otimes} B_2^{(I)}.$$

**Def** For factorization alg $B / X$, define its factorization homology as

$$H_\nabla(X, B) := C_{dR}(\mathrm{Ran}_X, B_{\mathrm{Ran}_X})$$
$$:= \operatorname*{colim}_I C_{dR}(X^I, B^{(I)})$$

**Thm** $H_\nabla(X, \mathcal{O})$ is trivial.

**Ex** Recall BD Grassmanian

$$Gr_I := \{ P \in Bun_G,\ (x_1, \ldots, x_I) \in X^I,\ \varphi : P|_{X \setminus \{x_i\}} \xrightarrow{\sim} P^{triv}|_{X \setminus \{x_i\}} \}.$$

$p_I : Gr_I \longrightarrow X^I$ & $Gr_I$ satisfy "factorization property"

can take $\boxed{B^{(I)} = p_{I*} \mathcal{O}_{Gr_I}}$

or more generally, $p_{I*}$ of any canonically defined sheaf

(e.g $\mathrm{Rep}(\hat{G})_{\mathrm{Ran}}$)

(qcoh pushforward is bad! Can do ind-coh or de Rham pushforward.)

we use this.

**Def** $B$ a fac. alg. A fac $B$-mod $M$ is a collection of D-mod $M^{\tilde{I}}$ on $X^{\tilde{I}}$, $\forall$ fin set $I$ w/

$\tilde{I} = I \cup \{*\}$, and $\forall\ \pi : \tilde{J} \longrightarrow \tilde{I}$ s.t $\pi(*) = *$,

have $X^{\tilde{I}} \xrightarrow{\Delta} X^{\tilde{J}}$ w/ isom $\Delta^* M^{\tilde{J}} \xrightarrow{\sim} M^{\tilde{I}}$

& $\forall$ partition $I = I_1 \sqcup \cdots \sqcup I_n$, fac. isom

$$M^{\tilde{I}}|_U \xrightarrow{\sim} \Big( \underset{0 \le i \le n}{\boxtimes} B^{I_i} \boxtimes M^{\tilde{I}_i} \Big)\Big|_U,\quad U \subseteq X^I \text{ open}$$

corresponding to the partition.

<u>Rmk</u> This also makes factorization module a monoidal cat.

## §2 Chiral Algebra

$$\begin{array}{ccccc} B^{(1)} \boxtimes B^{(1)} & & B^{(2)} & & B^{(1)} \\ U & \xhookrightarrow{j} & X \times X & \xleftarrow{\Delta} & X \end{array}$$

Factorization Alg $\longrightarrow$ Chiral Alg

$\Psi$

$B$ : Give each $B(I)$ a natural D-mod structure

$$B^{(I)} \boxtimes \mathcal{O}_{X^I} \xrightarrow{i_1} B^{(I \amalg I)} \xleftarrow{i_2} \mathcal{O}_{X^I} \boxtimes B^{(I)}$$

$\cong$

Gives D-mod structure!

$i_1, i_2$ are both isom over infinitesimal nbhd of diagonal

Let $A^{(I)} = B^{(I)} \otimes \omega_{X^I}$. Then $\Delta^! A^{(2)} = \Delta^* B^{(2)} \otimes \Delta^! \omega_{X^2}$

sits in deg. 0

$$= B^{(1)} \otimes \omega_X [-1]$$
$$= A^{(1)} [-1].$$

The triangle $\Delta_! \Delta^! A^{(2)} \longrightarrow A^{(2)} \longrightarrow j_* j^* A^{(2)}$

induces $A^{(2)} \longrightarrow j_* j^* (A^{(1)} \boxtimes A^{(1)}) \xrightarrow{\mu} \Delta_! A^{(1)}$ → Cousin complex

Records extension.
Called chiral bracket.

$\mu$ : anti symmetric : $\omega_{X^2} \xrightarrow{\sim} \omega_X \boxtimes \omega_X$

Cousin complex on $X^3$ :

$$A^{(3)} \longrightarrow j_* j^* A^{(3)} \longrightarrow \bigoplus_{i \neq j} \Delta_{x_i = x_j} j_* j^* A^{(2)} \longrightarrow \Delta_! A^{(1)}$$

$\|$

$j_* j^* (A^{(1)} \boxtimes A^{(1)} \boxtimes A^{(1)})$

Composition = Jacobi

<u>Def</u> <u>Chiral alg</u> is a D-mod $A$ over $X$ w/ a chiral bracket (antisymmetric & Jacobi id)

$\mu : j_* j^* (A \boxtimes A) \longrightarrow \Delta_! A$

& a map $\omega_X \longrightarrow A$ s.t restriction of $\mu$ to

$i_* j^* (\mathcal{A} \boxtimes \omega_X)$ is the one from the Cousin cplx:

$$0 \to \omega_X \boxtimes \mathcal{A} \to i_* j^* \omega_X \boxtimes \mathcal{A} \to \Delta_! \mathcal{A} \to \cdots$$

Def Chiral module...

Ex (1) $\mathcal{M} = \mathcal{A}$

(2) $\mathcal{M} = i_{x!} i_x^! \mathcal{A}[1]$, $i_{x!} i_x^! \mathcal{A} \to \mathcal{A}$

Thm $\exists$ equiv. of cat:

$$\{\text{chiral alg}\} \longleftrightarrow \{\text{Factorization alg}\}$$
$$\mathcal{A} \longleftrightarrow \mathcal{B}$$

& $\{\text{chiral } \mathcal{A}\text{-mod}\} \longleftrightarrow \{\text{fac. } \mathcal{B}\text{-mod}\}$

Break

Pf of Thm "$\leftarrow$" map already constructed.

"$\longrightarrow$": Let $\mathcal{A}$ = chiral algebra.

$$C_I^\bullet := (j_* j^* (\mathcal{A} \boxtimes \cdots \mathcal{A}) \to \bigoplus_{q \in P_{ort}^{|I|-1}(I)} \Delta_{q!} (j_* j^* \mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}) \to \cdots)$$

deg $= -|I|$ deg $= -|I|+1$

$\parallel$

Chevalley-Cousin complex.

but, connecting map of $C_I^\bullet$ given by alt. sum of chiral bracket.

Fact: $C_I^\bullet$ as complex satisfies factorization & !-pullback.

Claim $H^n(C_I^\bullet) = 0$ except $n = -|I|$. (Then, just take

$B^{(I)} = H^n(C_I^\bullet) \otimes \omega_{X^I}^{-1}$).

induction on $I = \{1, \underbrace{2, \dots, n}_{I'}\}$.

concentrated at deg $-|I|+1$

over $U$, is locally $C_{J_1}^\bullet \boxtimes C_{J_2}^\bullet$, $J_i \subsetneq I$, $J_1 \sqcup J_2 = I$.

$$X^I \xrightarrow[x_1 = x_2]{i} X^I \xleftarrow{v} U, \quad \overbrace{i_! C_{I'}^\bullet}^{} \to C_I \to v_* \underbrace{\overbrace{v^* C_I^\bullet}^{}}_{\text{deg } -|I|}.$$

Suffices : $H^{-|I|+1}(i_! C^\bullet_{I'}) \longrightarrow H^{-|I|+1}(C^\bullet_I)$ is $0$.

$$\begin{tikzcd} V_* V^*(\omega_X \boxtimes Z) \arrow[r] \arrow[d, "\text{chiral bracket}"'] & C_I^{-|I|} \arrow[d, "\text{chiral bracket}"] \\ i_! Z \arrow[r] & C_I^{-|I|+1} \end{tikzcd}$$

from Cousin cplx

where $Z = C^\bullet_{I'}$

Finishes the suffices ∎

# §3 Chiral Envelope

Def A Lie-* alg $L$ is a D-mod on $X$ w/ Lie bracket $\mu : L \boxtimes L \longrightarrow \Delta_!(L)$.

E.g. (1) $\mathfrak{g} \otimes \mathcal{O}_X$

(2) $\forall$ chiral alg $\mathcal{A}$, $\mathcal{A} \boxtimes \mathcal{A} \longrightarrow j_* j^* \mathcal{A} \boxtimes \mathcal{A} \xrightarrow{\mu} \Delta_! \mathcal{A}$ {Lie-* alg}

Thm $\exists$ left adjoint {Lie-* alg} $\longrightarrow$ {chiral alg}

$L \rightarrow A(L)$ to the above functor called chiral envelope.

(So, $\mathrm{Hom}_{\text{Lie-* alg}}(L, \mathcal{A}) = \mathrm{Hom}_{\text{chiral alg}}(A(L), \mathcal{A})$). (*)

pf sketch

$$\begin{tikzcd} U \arrow[r, hook, "j"] & X^I \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X^I & & X \end{tikzcd}$$

$\tilde{L}_0^{(I)} := p_{1*} p_2^*(L) \otimes \omega_X^{-1}$

$\tilde{L}^{(I)} := p_{1*} j_* j^* p_2^*(L) \otimes \omega_{X^I}^{-1}$

$B(L)^{(I)} := U(\tilde{L}^{(I)}) / U(\tilde{L}_0^{(I)}) = \mathrm{Ind}_{\tilde{L}_0^{(I)}}^{\tilde{L}^{(I)}}(\mathcal{O}_{X^I})$

$$0 \to \tilde{L}_0^{(1)} \to \tilde{L}^{(1)} \to L \otimes \omega_X^{-1} \to 0$$

induces $L \otimes \omega_X^{-1} \longrightarrow B(L)^{(1)}$ by embedding in deg 1.

Pf (*) (left-adjointness):

$$j_* j^* (L \boxtimes A) \longrightarrow \Delta_! A,$$

apply $P_{2*} \rightsquigarrow \tilde{L}^{(1)} \to B^{(1)}$

$$\rightsquigarrow U(\tilde{L}^{(1)}) \longrightarrow B^{(1)}$$

And also prove $U(\tilde{L}_0^{(1)}) \to \mathcal{O}$ on unit in $B^{(1)}$

Hence have a map $B(L)^{(1)} \to B^{(1)}$ ∎

E.g. $L = \mathfrak{g} \otimes \mathcal{D}_X$

$$A(L)_x = \mathrm{Ind}_{H_{dR}(X, \mathfrak{g} \otimes D_X)}^{H^0_{dR}(X \setminus x, \mathfrak{g} \otimes D_X)} \mathbb{C} = \mathrm{Ind}_{H^0_{dR}(D, \mathfrak{g} \otimes D_X)}^{H^0_{dR}(D^\circ, \mathfrak{g} \otimes D_X)} \mathbb{C}$$

stalk

$$= \mathrm{Ind}_{\mathfrak{g}(\mathcal{O}_x)}^{\mathfrak{g}(K_x)} \mathbb{C} \simeq \mathrm{Vac} \quad \text{(up to central extension)}$$

Thm $H_\nabla(A(L)) \simeq C_\bullet(C_{dR}(X, L))$

$C_\bullet$: homo. chev. complex, computing derived functor of taking coinvariants.

$C_{dR}(X, L)$: Lie alg. in Vect

## §4 Comm. Chiral Alg

**Def** Say $A$ is commutative chiral alg if composition

$$A \boxtimes A \longrightarrow j_* j^* (A \boxtimes A) \xrightarrow{\mu} \Delta_! A$$

vanishes.

Equivalently, $\mu$ factors thru $\Delta_! \underbrace{\Delta^!(A \boxtimes A)}_{\| A \otimes^! A}$.

**Prop** $\{\text{comm. chiral alg}\} \xrightarrow{\sim} \{\text{comm. left } \mathcal{D}_X\text{-mod}\}$

$$A \longmapsto A \otimes \omega_X^{-1}$$

**Thm** Let $A$ be chiral alg w/ compatible unital binary operation $m : A \overset{!}{\otimes} A \to A$.

Then $A$ is comm. & $\mu$ "=" $m$.

**Idea** In usual alg, such an operation is a vertical prod:

$$ac - bd = \begin{matrix} a & - & b \\ | & & | \\ c & - & d \end{matrix} = \begin{matrix} ab \\ | \\ cd \end{matrix}$$

Then unit coincide

$$\begin{matrix} 1_{\text{vert}} & - & 1_{\text{hor}} \\ | & & | \\ 1_{\text{hor}} & - & 1_{\text{vert}} \end{matrix}$$

and

$$ab = \begin{matrix} 1 & - & b \\ | & & | \\ a & - & 1 \end{matrix} = \begin{matrix} b & - & 1 \\ | & & | \\ 1 & - & a \end{matrix} = ba$$

**pf** copy.

$$\begin{tikzcd} j_* j^* (A \overset{!}{\otimes} A) \boxtimes (A \overset{!}{\otimes} A) \arrow[r, "\mu \boxtimes \mu"] \arrow[d] & \Delta_! (A \overset{!}{\otimes} A) \arrow[d, "m)"] \\ j_* j^* (A \boxtimes A) \arrow[r, "\mu"] & \Delta_! (A) \end{tikzcd}$$

$\alpha, \beta : \omega_X \to A$ unit for $\mu$ & $m$. prove $\omega_\alpha = \omega_\beta$ by looking at $(\omega_\alpha \overset{!}{\otimes} \omega_\beta \boxtimes \omega_\alpha \overset{!}{\otimes} \omega_\beta)$ $\blacksquare$

**Rmk** • Can be used to show comm. chiral alg is given by a factorization alg of global section of multi-jets space.

comm. fact. alg is given by functions on horizontal jets (full jet space is $\infty$-dim) $\Rightarrow$ fin. dim fibers.

• If $\mathcal{A}$ is comm. chiral alg, $\begin{array}{c} \operatorname{Spec} \mathcal{B} \\ \downarrow f \\ X_{dR} \end{array}$,

then chiral homology $= \Gamma(\operatorname{Sect} of f)$

• Eckman-Hilton also used to show $\operatorname{End}(\operatorname{Vac})$ is commutative